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Functions with Non-Negative Convolutions

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1. INTRODUCTION

Let $\mathscr{G}$ be a group and $\mathscr{F}$ a class of real-valued functions on $\mathscr{G}$ such that if φ and ψ are in $\mathscr{F}$, then their convolution $\varphi * \psi$ is also in $\mathscr{F}$. Given $f \in \mathscr{F}$, we shall investigate whether there exists a non-negative function $g \in \mathscr{F}$, $g \not\equiv 0$, such that the convolution $f * g$ is non-negative on $\mathscr{G}$. We let $\mathscr{P}(f) = \mathscr{P}(f, \mathscr{F})$ denote the (possibly empty) set of functions g described above. If there is one element $g \in \mathscr{P}(f)$, then there are generally many, since $g * \varphi \in \mathscr{P}(f)$ for any non-negative $\varphi \in \mathscr{F}$.

This study originated from consideration of the Wiener tauberian theorem for functions satisfying a one-sided bound. See e.g., Wiener [11, Theorem XI.] Heretofore, non-negativity of the "kernel" has been a necessary condition for such theorems. If the kernel f changes sign but there exists a non-negative g such that $f * g$ is non negative, then the classical argument may be valid under certain conditions with $f * g$ in place of f. We shall give such applications in a separate article to appear in *Proc. London Math. Soc.* (3) **36** (1978).

As a simple example, we consider the case $G = \mathscr{T}$, the circle group, and $\mathscr{F} = L^1(\mathscr{T})$. Let $\hat{f}$ denote the Fourier transform of f.

THEOREM 1.1. *Let $f \in L^1(\mathscr{T})$. There exists a function $g \in \mathscr{P}(f)$ if and only if*

$$\hat{f}(0) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(x)\, dx \geqslant 0.$$

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Proof. If $\hat{f}(0) \geqslant 0$, take $g \equiv 1$ and note that $f * g(x) = \hat{f}(0) \geqslant 0$. Conversely, if there exists a $g \in \mathscr{P}(f)$, we have

$$0 \leqslant (f * g)\hat{}(0) = \hat{f}(0)\, \hat{g}(0).$$

Since $g \geqslant 0$ and $g \not\equiv 0$, it follows that $\hat{g}(0) > 0$ and hence $\hat{f}(0) \geqslant 0$.

This theorem has an obvious analogue for almost periodic functions on $\mathscr{R}$.

The simplicity of these results derives from the fact that constants are eligible functions in the space of periodic or almost periodic functions. Henceforth we shall study our problem for the groups $\mathscr{R}^d$ and $\mathscr{Z}^d$, $d \geqslant 1$, and the following classes of real-valued measurable functions.

1. Let $q\colon \mathscr{R}^+ \to \mathscr{R}$ be a non-decreasing function satisfying
   (i) $q(t) \uparrow \infty$, $t \to \infty$,
   (ii) $q(t) = O(t)$, $t \to \infty$.

Let $\mathscr{F}_q$ consist of all measurable functions on $\mathscr{R}^d$ or $\mathscr{Z}^d$ satisfying

$$f(x) = O(\exp(-q(a \mid x \mid))), \quad x \to \infty,$$

for some positive number $a = a(f)$. Here $| \cdot |$ denotes the Euclidean norm.

We note that $V(\cdot) = \exp(-q(\cdot))$ satisfies

$$\liminf_{t\to\infty} \frac{\log V(t)}{t} > -\infty,$$

and we describe this situation by saying that V is of *subexponential decay.*

2. Let $\mathscr{F}_0$ consist of measurable functions f on $\mathscr{R}^d$ or $\mathscr{Z}^d$ satisfying

$$\lim_{|x|\to\infty} (\log |f(x)|)/| x | = -\infty.$$

We say that functions in $\mathscr{F}_0$ are of *superexponential decay.*

3. Let $\mathscr{F}_c$ consist of measurable functions on $\mathscr{R}^d$ or $\mathscr{Z}^d$ having compact support.

Our main results in these three cases can be summarized as follows. Let $x = (x_1, ..., x_d)$ and $t = (t_1, ..., t_d)$ denote elements of $\mathscr{R}^d$ or $\mathscr{Z}^d$, let $xt = \sum_{i=1}^{d} x_i t_i$, and let

$$Lf(t) = \int_{\mathscr{G}} f(x)\, e^{xt}\, dx$$

denote the Laplace transform of f, where dx denotes Haar measure on $\mathscr{G}$. Then

1. $\mathscr{P}(f, \mathscr{F}_q) \neq \varnothing \Leftrightarrow Lf(0) > 0$;
2. $\mathscr{P}(f, \mathscr{F}_0) \neq \varnothing \Leftrightarrow Lf(t) > 0,\ \forall t \in \mathscr{G}$;
3. for $f \in \mathscr{F}_c$ and f boundary-definite (defined in sections 4 and 6), $\mathscr{P}(f, \mathscr{F}_c) \neq \varnothing \Leftrightarrow Lf(t) > 0,\ \forall t \in \mathscr{G}$.

If $\mathscr{G} = \mathscr{Z}$, the boundary-definite condition is inessential. On the other hand, we shall later give an example of a compactly supported function $f \in L^1(\mathscr{R})$ for which $Lf(t) > 0$, $\forall t \in \mathscr{R}$, and $\mathscr{P}(f, \mathscr{F}_c) = \varnothing$.

For $\mathscr{F} = \mathscr{F}_c(\mathscr{Z})$ we obtain a new proof of an old result of Poincaré [7] and Meissner [6] characterizing polynomials which can be expressed as quotients of polynomials with positive coefficients. For $\mathscr{F} = \mathscr{F}(\mathscr{Z}^d)$, $d > 1$, these results have been extended to polynomials in d variables by Hardy, Littlewood and Pólya ([4], 2.24). Our general method might be used to study analogous problems in $\mathscr{R}^d$, $d > 1$ (see Section 6).

Finally, we mention another result which bears some relation to ours. Define an exponential polynomial to be a function of the form $\sum_{k=1}^{n} a_k e^{b_k z}$, where the $\{a_k\}$ and $\{b_k\}$ are sequences of complex numbers. It was shown by Ritt [9] that if the quotient of two exponential polynomials is an entire function, then the quotient itself is an exponential polynomial. For further results of this type, see e.g., Shields [10] and Berenstein and Dostal [2a, b].

2. Subexponential Decay

The case $\mathscr{F} = \mathscr{F}_q$. There are essential differences in the theory according to whether all functions in the class $\mathscr{F}$ vanish at infinity faster than any exponential or if this is not the case. In the present section we consider the latter alternative.

Let $q: \mathscr{R} \to \mathscr{R}$ be a continuous function satisfying

$$q(0) = 0, \tag{2.1}$$

$$q \text{ is even}, \tag{2.2}$$

$$q'(x) > 0,\ x > 0, \tag{2.3}$$

$$q'' \text{ is continuous on } \mathscr{R} - \{0\} \text{ and } q''(x) \leqslant 0 \text{ for } x \neq 0, \tag{2.4}$$

there exists a constant $c_0 \in (0, 1)$ such that

$$\int_{-\infty}^{\infty} \exp\{-q(x) + q(c_0 x)\}\, dx < \infty. \tag{2.5}$$

These hypotheses imply that $q(x) \geqslant 0$ and $\lim_{|x|\to\infty} q(x) = \infty$. Also, q is subadditive, i.e.

$$q(x + y) \leqslant q(x) + q(y),\ x, y \in \mathscr{R}. \tag{2.6}$$

Indeed, in each of the three cases (i) $xy \geqslant 0$, (ii) $xy < 0$ and $|x + y| \leqslant |x|$, (iii) $xy < 0$ and $|x + y| > |x|$, we have

$$q(x + y) - q(x) \leqslant \int_{|x|}^{|x|+|y|} q'(t)\, dt \leqslant \int_0^{|y|} q'(t)\, dt = q(y).$$

Further, we note that

$$\frac{d^2}{dx^2}(\exp\{-q(x)\}) > 0, \qquad x \neq 0. \tag{2.7}$$

For q satisfying (2.1)–(2.5) we define $\mathscr{F}_q$ as in Section 1. As examples we mention

(i) $q(x) = \text{const.}\ |x|^\alpha,\ 0 < \alpha \leqslant 1.$

(ii) $q(x) = \text{const.}\ \{\log^\beta(e^\beta + |x|) - \beta^\beta\},\ \beta \geqslant 2.$

We remark that one can replace (2.5) by less restrictive conditions, but our arguments would be somewhat more complicated.

THEOREM 2.1. *Let q satisfy conditions* (2.1)–(2.5). *Let* $f \in \mathscr{F}_q(\mathscr{R})$, $f \not\equiv 0$. *Then the following are equivalent.*

A. $\int f(x)\, dx > 0.$

B. *There exists* $\epsilon_0 > 0$ *such that the function* $x \to g_\epsilon(x) = \exp(-q(\epsilon x))$ *satisfies* $f * g_\epsilon \geqslant 0$ *on* $\mathscr{R}$ *for* $0 < \epsilon \leqslant \epsilon_0$.

C. *There exists an* $\epsilon > 0$ *such that* $f * g_\epsilon \geqslant 0$ *on* $\mathscr{R}$.

Proof. $A \Rightarrow B$. Let ϵ and a be positive numbers to be specified later. We form

$$(f * g_\epsilon)(x) = \left\{\int_{|u|<a} + \int_{|u|>a}\right\} f(u) \exp\{-q(\epsilon x - \epsilon u)\}\, du = I_1 + I_2,$$

say.

If we set

$$M(\epsilon, a) = \sup |\exp\{q(\epsilon x) - q(\epsilon x - \epsilon u)\} - 1|,\ |u| \leqslant a,\ x \in \mathscr{R},$$

it follows from (2.6) that

$$M(\epsilon, a) \leqslant \exp(q(\epsilon a)) - 1 = o(1),\ (\epsilon a \to 0).$$

We have

$$\begin{aligned} I_1 &= e^{-q(\epsilon x)} \left\{\int_{-a}^{a} f(u)\, du + \int_{-a}^{a} f(u)(\exp\{q(\epsilon x) - q(\epsilon x - \epsilon u)\} - 1)\, du\right\} \\ &\geqslant e^{-q(\epsilon x)} \left\{\int_{-a}^{a} f(u)\, du - M(\epsilon, a) \int_{-a}^{a} |f(u)|\, du\right\}. \end{aligned}$$

We estimate I_2 with the aid of (2.6) and the bound $f(u) = o\{\exp(-q(bu))\}$ given in the definition of $\mathscr{F}_q$ in Section 1:

$$\begin{aligned} |I_2| &\leqslant \exp(-q(\epsilon x)) \int_{|u|>a} |f(u)| \exp(q(\epsilon u))\, du \\ &\leqslant \exp(-q(\epsilon x)) \int_{|u|>a} K \exp\{-q(bu) + q(\epsilon u)\}\, du \\ &= \exp(-q(\epsilon x))\, (K/b) \int_{|v|>ab} \exp\{-q(v) + q(\epsilon v/b)\}\, dv. \end{aligned}$$

The last integral is convergent by (2.5) provided that $\epsilon/b \leqslant c_0$. We assume henceforth that $\epsilon \leqslant c_0 b$.

Combining these estimates we obtain

$$(f * g_\epsilon)(x) \exp(q(\epsilon x))$$
$$\geqslant \int_{-a}^{a} f(u)\, du - M(\epsilon, a) \int_{-\infty}^{\infty} |f(u)|\, du - (K/b) \int_{|v|>ab} \exp\{-q(v) + q(c_0 v)\}\, dv$$

Now choose a so large that

$$\int f(u)\, du < 2 \int_{-a}^{a} f(u)\, du,$$
$$(K/b) \int_{|v|>ab} \exp(-q(v) + q(c_0 v))\, dv < \left(\int f(u)\, du\right)\Big/8.$$

Next choose $\epsilon_0 \leqslant c_0 b$ and so small that

$$M(\epsilon_0, a) \int |f(u)|\, du < \left(\int f(u)\, du\right)\Big/8.$$

With these choices of ϵ_0 and a, we have

$$f * g_\epsilon(x) \geqslant \exp\{-q(\epsilon x)\} \int f(u)\, du/4$$

for any ϵ in $(0, \epsilon_0]$. Thus $A \Rightarrow B$.

$B \Rightarrow C$. Obvious.

$C \Rightarrow A$. We have

$0 \leqslant \int f * g_\epsilon(x)\, dx = (\int f(x)\, dx)(\int g_\epsilon(x)\, dx)$. Since $\int g_\epsilon(x)\, dx > 0$, it follows that $\int f(x)\, dx \geqslant 0$. If $\int f(x)\, dx = 0$, then $\int f * g_\epsilon(x)\, dx = 0$. Since $f * g_\epsilon \geqslant 0$, it follows that $f * g_\epsilon = 0$ a.e. Taking Fourier transforms we obtain $\hat{f}(t)\, \hat{g}_\epsilon(t) = 0$, $t \in \mathscr{R}$. Since g_ϵ is even, we can express $\hat{g}_\epsilon$ as a cosine transform. Arguing in a standard way, we perform two integrations by parts, recall (2.7) and obtain for $t \neq 0$

$$\hat{g}_\epsilon(t) = 2 \int_0^{\infty} t^{-2}(1 - \cos xt)\, g_\epsilon''(x)\, dx > 0. \tag{2.8}$$

It follows that $\hat{f} = 0$ a.e., and hence $f = 0$ a.e., which contradicts a hypothesis. Thus $\int f(x)\, dx > 0$.

Theorem 2.1 has an analogue for higher dimensional Euclidean space $\mathscr{R}^d$. We require the following extension of condition (2.5) to enable us to treat a convolution integral:

There exists a constant $c \in (0, 1)$ such that

$$\int_0^{\infty} t^{d-1} \exp\{-q(t) + q(ct)\}\, dt < \infty. \tag{2.5$'$}$$

This condition can be deduced from (2.5) in many cases, as we now show.

LEMMA 2.1. *Let q satisfy conditions* (2.1)–(2.5), *let d be a positive integer, and suppose* (2.5) *holds for some $c_0 > d/(d+1)$. Then* (2.5′) *holds.*

Proof. Since q' is a decreasing function on $(0, \infty)$ we have for $t > 0$ and $0 < c < 1$

$$(1-c)\, tq'(t) \leqslant q(t) - q(ct) \leqslant (1-c)\, tq'(ct).$$

This inequality and (2.5) imply that

$$\int_0^\infty \exp\{-\beta_0 tq'(t)\}\, dt < \infty,$$

where $\beta_0 = (1-c_0)/c_0$. Thus,

$$\int_A^\infty \exp\{-\beta_0 tq'(t)\}\, dt \to 0, \qquad A \to \infty.$$

Let $\epsilon > 0$ be given. For $A = A(\epsilon)$ sufficiently large, we have

$$\epsilon A \exp\{-\beta_0(1+\epsilon)\, Aq'(A)\} < \int_A^{A+\epsilon A} \exp\{-\beta_0 tq'(t)\}\, dt < \epsilon.$$

It follows that

$$\log t < (1+\epsilon)\beta_0 \;\; tq'(t)$$

for all $t \geqslant A(\epsilon)$. Now for $0 < c < c_0$ and $t \geqslant A$,

$$\begin{aligned}
&\int_A^\infty t^{d-1} \exp\{-q(t) + q(ct)\}\, dt \\
&\qquad \leqslant \int_A^\infty \exp\{(d-1)\log t - (1-c)\, tq'(t)\}\, dt \\
&\qquad < \int_A^\infty \exp\{(d-1-(1-c)/\{\beta_0(1+\epsilon)\})\log t\}\, dt.
\end{aligned}$$

The last integral is finite provided that $d < (1-c)\, c_0/((1+\epsilon)(1-c_0))$. However, we have assumed that $d < c_0(1-c_0)^{-1}$. Thus there exist small positive numbers ϵ and c such that the penultimate inequality holds for d. This completes the proof of Lemma 2.1.

For example, let

$$q(t) = K\{\log^2(e^2 + |\, t\,|) - 4\}$$

and let d be a positive integer. Then (2.5′) holds provided that $K > 0$.

We define $Q(x) = q(|\, x\,|)$, where q satisfies (2.1)–(2.4) and (2.5′) and $|\, x\,|$ denotes the Euclidean norm of x in $\mathscr{R}^d$. We have

THEOREM 2.2. *Let q and Q be as above and let $f \in \mathscr{F}_Q(\mathscr{R}^d)$, $f \not\equiv 0$. Then the following are equivalent.*

A. $\int f(x)\,dx > 0$.

B. *There exists $\epsilon_0 > 0$ such that the function $x \to g_\epsilon(x) = \exp(-Q(\epsilon x))$ satisfies $f * g_\epsilon \geqslant 0$ on $\mathscr{R}^d$ for $0 < \epsilon \leqslant \epsilon_0$.*

Proof. The assertion $A \Rightarrow B$ is established as before. For the converse, we again show that $\int f(x)\,dx \geqslant 0$, assume equality, and deduce a contradiction.

As before $\int f(x)\,dx = 0$ implies that $\hat{f}(x)\,\hat{g}_\epsilon(x) = 0$ for $0 < \epsilon \leqslant \epsilon_0$. Since we assume that $f \not\equiv 0$ on $\mathscr{R}^d$, there exists $x_0 \in \mathscr{R}^d$ for which $\hat{f}(x_0) \neq 0$. It follows that

$$\hat{g}_\epsilon(x_0) = \epsilon^{-d}\hat{g}(x_0/\epsilon) = 0, \qquad 0 < \epsilon \leqslant \epsilon_0 .$$

This in turn implies that $\hat{g}$ is compactly supported, since $\hat{g}_\epsilon$ is radially symmetric. Thus g is the restriction to $\mathscr{R}^d$ of an entire function. Furthermore, the function $h\colon \mathscr{R} \to \mathscr{R}$ defined by

$$h(y) = g(y, 0, 0,..., 0) = \exp(-q(y)),$$

is the restriction to $\mathscr{R}$ of an entire function. We have $h'(0) = 0$ since $h(y) = h(-y)$ for any $y \in \mathscr{R}$. For $y > 0$ we have

$$h'(y) = h'(y) - h'(0) = yh''(\xi)$$

for some $\xi > 0$. It follows from (2.7) that $h' > 0$ on $\mathscr{R}^+$. On the other hand (2.3) yields

$$h'(y) = e^{-q(y)}(-q'(y)) < 0, \qquad y > 0.$$

This contradiction implies that $\int f(x)\,dx > 0$.

Similar results in $\mathscr{R}^d$, $d \geqslant 1$, can also be established for classes $\mathscr{F}_q$ which are such that if $f \in \mathscr{F}_q$, then the Fourier transform $\hat{f}$ is quasi-analytic. A quasi-analytic function has the property that if the function vanishes on an open set, then it must vanish identically. If $f \in \mathscr{F}_q$, $\hat{f}$ has this property only if

$$\int_{-\infty}^{\infty} (1 + s^2)^{-1}\, q(s)\, ds = \infty.$$

This condition requires that q grows nearly linearly, and is thus rather more stringent than (2.5).

It is easy to establish the analogue of Theorem 2.1 for real-valued functions on $\mathscr{Z}$ and the analogue of $A \Rightarrow B$ in Theorem 2.2. for real-valued functions on $\mathscr{Z}^d$, $d > 1$. We replace integrals by sums, derivatives by differences, and Fourier integrals by Fourier series.

It is somewhat more complicated to find an analogue of $B \Rightarrow A$ in Theorem 2.2 for real-valued functions on $\mathscr{Z}^d$, $d > 1$. One possibility is to use the aforementioned quasi-analyticity hypothesis. This guarantees that the Fourier series $\sum g_\epsilon(n)e^{inx}$ does not vanish on an open set and the proof is easily completed.

The quasi-analytic condition is rather restrictive. We can avoid it by applying Theorem 2 in [1a]. We are grateful to Professor R. Askey for bringing this result to our attention. Re-stated in a way which fits our purposes, it is as follows.

THEOREM A. *Let* $G(x) = \gamma(|x|)$ *be a continuous, radial function on* $\mathscr{R}^d$. *If*

(i) $\gamma(t) \to 0$, $t \to \infty$,

(ii) $(-1)^{[d/2]}\gamma^{[d/2]}(t)$ *is convex,* $t \geqslant 0$, *then* G *is positive definite, i.e., it is the Fourier transform of a non-negative measure. If we furthermore assume that* $G \in L^1(\mathscr{R}^d)$, *then* $\hat{G}(x) > 0$ *for all* $x \in \mathscr{R}^d$.

It is clear from Askey's proof [1b] that Theorem A is true. We note in particular that it follows from conditions (i) and (ii) of Theorem A that we have

$$(-1)^k\gamma^{(k)}(t) \geqslant 0, \quad t \geqslant 0, \quad k = 0, 1,..., [d/2].$$

This observation is due to Levy.

Our next step is given by

LEMMA 2.2. *Let* $G \in L^1(\mathscr{R}^d)$ *be as in Theorem A and define* $g(n) = G(n)$, $n \in \mathscr{Z}^d$. *If* $f \in L^1(\mathscr{Z}^d)$, $f \not\equiv 0$, *and* $f * g \geqslant 0$, *then we have*

$$\hat{f}(0) = \sum_{n\in\mathscr{Z}^d} f(n) > 0. \tag{2.9}$$

Proof. Using the convexity properties of G and the fact that $G \in L^1(\mathscr{R}^d)$, we see that $g \in L^1(\mathscr{Z}^d)$. According to the Poisson summation formula, it follows from Theorem A that

$$\hat{G}(x) = \sum_{n\in\mathscr{Z}^d} G(n)\, e^{inx} = \sum_{m\in\mathscr{Z}^d} \hat{G}(2\pi m - x) > 0.$$

Arguing as in the proof $C \Rightarrow A$ in Theorem 2.1, we obtain (2.9). The lemma is proved.

Let q satisfy the conditions of Theorem 2.2 and let $G(x) = \exp\{-q(\epsilon \mid x \mid)\}$ satisfy the conditions of Theorem A. (One possibility is to take $q(x) = \text{const.}\ |x|^\alpha$, $0 < \alpha \leqslant 1$.) We also define $g_\epsilon(n) = G(n)$, $n \in \mathscr{Z}^d$. Using Lemma 2.2, it is now easy to see that the analogue of $B \Rightarrow A$ in Theorem 2.2 also holds on $\mathscr{Z}^d$, $d > 1$, with $f \in \mathscr{F}_q(\mathscr{Z}^d)$ and g_ϵ defined as above.

3. Superexponential Decay

Let $f \in \mathscr{F}_0(\mathscr{R}^d)$, i.e., f vanishes at ∞ faster than any exponential. It is clear that the Laplace transform Lf converges absolutely for all real or complex arguments. It can occur that Lf is positive on $\mathscr{R}^d$ but $\inf\{Lf(x): x \in \mathscr{R}^d\} = 0$. The following lemma will enable us to treat this situation.

LEMMA 3.1. *Let $F: \mathscr{R}^d \to \mathscr{R}^+$ be continuous but otherwise arbitrary. There exists a non-negative function G on $\mathscr{R}^d$, which is continuous on $\mathscr{R}^d$, of superexponential decay, and satisfies $LG \geqslant F$ on $\mathscr{R}^d$.*

Proof. We first consider the one-dimensional case. Define $\varphi: \mathscr{R} \to \mathscr{R}$ by $\varphi(x) = \max_{|t| \leqslant x} F(t)$. φ is a positive, even and continuous function, non-decreasing on $(0, \infty)$. We shall define $G(t) = 0$ for $|t| \leqslant \beta/2$, $G(t)$ is linear for $\beta/2 \leqslant |t| \leqslant \beta$, and $G(t) = K \exp\{-|t|\, h(t)\}$ for $|t| \geqslant \beta$. Here K and β are positive constants and h is an even function which tends monotonically to ∞ on $(0, \infty)$. These quantities will be specified later. By symmetry it suffices to show that $LG(x) \geqslant \varphi(x)$ for $x \geqslant 0$.

For $x \geqslant 3$ we estimate $LG(x)$ where the integrand is large. Set

$$I_x = \{t > 0 : x/3 \leqslant h(t) \leqslant 2x/3\}.$$

On I_x we have

$$\exp\{t(x - h(t))\} \geqslant \exp\{h^{-1}(x/3)x/3\} \geqslant h^{-1}(x/3).$$

Let $|I|$ denote the length of an interval I. If $(d/dy)\, h^{-1}(y) \geqslant 1$ for $y \geqslant 1$, then

$$|I_x| = h^{-1}(2x/3) - h^{-1}(x/3) \geqslant x/3 \geqslant 1.$$

Thus $LG(x) \geqslant h^{-1}(x/3)$ for $x \geqslant 3$. Now take h^{-1} on $[0, \infty)$ to be a function with derivative at least 1 and such that $h^{-1}(x) \geqslant \varphi(3x)$, e.g.

$$h^{-1}(x) = x + \int_x^{x+1} \varphi(3t)\, dt.$$

We have $LG(x) \geqslant \varphi(x)$ for $x \geqslant 3$.

Now choose $K \geqslant 1$ and large enough so that

$$K \int_\beta^\infty e^{-th(t)}\, dt \geqslant \varphi(3),$$

where $\beta = h^{-1}(0)$. For $0 \leqslant x < 3$ we have

$$LG(x) \geqslant K \int_\beta^\infty e^{-th(t)}\, dt \geqslant \varphi(3) \geqslant \varphi(x).$$

We note that G, as so defined, is continuous on $\mathscr{R}$ and vanishes superexponentially at infinity.

For $\mathscr{R}^d$ with $d \geqslant 2$ we use a similar argument. For simplicity we give the details for $d = 2$. Let

$$\varphi(x) = \varphi(|\, x \,|) = \max_{|t| \leqslant x} F(t).$$

Let G_1 be a non-negative valued radially symmetric function on $\mathscr{R}^2$, presently to be specified. We have

$$\begin{aligned} LG_1(x) &= \int_0^{2\pi} \int_0^\infty pG_1(p) \exp\{|\, x \,|\, p \cos \varphi\}\, dp\, d\varphi \\ &\geqslant \frac{2\pi}{3} \int_0^\infty pG_1(p) \exp\{|\, x \,|\, p/2\}\, dp, \qquad x \in \mathscr{R}^2. \end{aligned}$$

From the case $d = 1$ we can construct a non-negative even function G on $\mathscr{R}$ of superexponential decay satisfying $LG(x) \geqslant \varphi(2\,|\, x \,|)$. Choosing

$$G_1(y) = (3/\pi)\, G(|\, y \,|)/|\, y \,|, \qquad y \in \mathscr{R}^d,$$

we obtain $LG_1(x) \geqslant \varphi(|\, x \,|)$. We note that G_1 is continuous since G is continuous and vanishes in a neighborhood of the origin.

Given $f \in \mathscr{F}_0(\mathscr{R}^d)$, the results of Section 2 show that if $\hat{f}(0) > 0$, then there exists a positive function g, of exponential rate of decay at infinity, for which $f * g \geqslant 0$. In the following theorem we give a necessary and sufficient condition on f in order that there exists a $g \in \mathscr{P}(f, \mathscr{F}_0)$,

THEOREM 3.2. *Let $f \in \mathscr{F}_0(\mathscr{R}^d)$, $f \not\equiv 0$. Then the following conditions are equivalent.*

A. *There exists a continuous, non-zero, non-negative function g on $\mathscr{R}^d$, of superexponential decay, satisfying $f * g \geqslant 0$ on $\mathscr{R}^d$.*

B. *$Lf > 0$ on $\mathscr{R}^d$.*

Proof. Suppose condition A. holds. Arguing as before, we have

$$0 \leqslant L(f * g) = Lf \cdot Lg.$$

Since $Lg > 0$, it follows that $Lf \geqslant 0$. If $Lf(x_0) = 0$ for some $x_0 \in \mathscr{R}^d$, then $L(f * g)(x_0) = 0$. Since $f * g \geqslant 0$ we have $L(f * g) = 0$ on $\mathscr{R}^d$. It follows that $Lf = 0$ on $\mathscr{R}^d$. By the uniqueness theorem for Laplace transforms $f = 0$ on $\mathscr{R}^d$, contradicting a hypothesis.

Now suppose condition B. holds. We claim that it suffices to prove the result under the stronger hypothesis that $Lf \geqslant 1$ on $\mathscr{R}^d$. Indeed, suppose we are given $h: \mathscr{R}^d \to \mathscr{R}$ with $Lh > 0$ on $\mathscr{R}^d$. Applying Lemma 3.1 to the function $F(x) = (Lh(x))^{-1}$ we obtain a non-negative function G on $\mathscr{R}^d$ of superexponential decay and satisfying $LG \geqslant F$. The function $h * G$ has superexponential decay, since it is the convolution of two functions having this property, and it satisfies $L(h * G) = Lh \cdot LG \geqslant 1$. By the special case of the theorem (to be proved) there exists a non-negative valued function g on $\mathscr{R}^d$, $g \not\equiv 0$, having super-exponential decay, for which $(h * G) * g \geqslant 0$ on $\mathscr{R}^d$. The function $G * g \in \mathscr{P}(h, \mathscr{F}_0)$. Thus it suffices to prove $B \Rightarrow A$ under the assumption $Lf \geqslant 1$. Let $e(x) = x/|x|$, $x \neq 0$.

LEMMA 3.3. *Let $x, y \in \mathscr{R}^d$, $x \neq 0$. Then*

$$|\, xy/|x| - |x| \,| \leqslant |x - y|.$$

Proof. This follows from the inequalities

$$|x|\,|x-y| + xy \geqslant x(x-y) + xy = |x|^2,$$
$$|x|\,|x-y| - xy \geqslant -x(x-y) - xy = -|x|^2.$$

LEMMA 3.4. *Let M: $[0, \infty) \to [0, \infty)$ be a continuous non-decreasing function. Let g: $\mathscr{R}^d \to \mathscr{R}^+$ satisfy*

$$g(x) = \exp\left\{- \int_0^{|x|} M(u)\, du\right\}.$$

Then for all $x, y \in \mathscr{R}^d$, $x \neq 0$,

$$g(x-y) \leqslant g(x) \exp\{ye(x)\, M(|x|)\}.$$

Proof. We first note that

$$g(x-y) = g(x) \exp\left\{- \int_{|x|}^{|x-y|} M(u)\, du\right\}.$$

Thus it suffices to show that

$$\int_{|x-y|}^{|x|} M(u)\, du \leqslant M(x)\, ye(x), \qquad x \neq 0. \tag{3.1}$$

By the monotonicity of M and Lemma 3.3 we obtain (for all values of $|x|$ and $|x-y|$)

$$\begin{aligned}\int_{|x-y|}^{|x|} M(u)\, du &\leqslant M(|x|)\,(|x| - |x-y|) \leqslant M(|x|)\,(|x| - \{|x| - ye(x)\}) \\ &= M(|x|)\, ye(x).\end{aligned}$$

This establishes (3.1) and hence the lemma.

By assumption there exists a non-decreasing unbounded function h: $\mathscr{R}^+ \to \mathscr{R}^+$ for which

$$f(x) = O(\exp\{-|x|\,(1 + h(|x|))\}), \qquad |x| \to \infty.$$

We shall construct $g \in \mathscr{P}(f, \mathscr{F}_0)$ of the form

$$g(x) = \exp\left\{-\int_0^{|x|} M(t)\,dt\right\}, \qquad x \in \mathscr{R}^d,$$

where $M\colon [0, \infty) \to [0, \infty)$ is differentiable, non-decreasing, and unbounded.

In the construction we shall require another non-negative, continuous, non-decreasing, unbounded function q on $[0, \infty)$ such that q and M satisfy the following conditions:

$$M(t) \leqslant h(q(t)), \qquad t > 0. \tag{3.2}$$

$$M'(s) \leqslant t^{-1}(1 + M(t)),\ |s - t| \leqslant q(t),\ t \geqslant t_0 > 0. \tag{3.3}$$

$$q(t)(M(t) + 1) \leqslant \tfrac{1}{2}\log(t + 100), \qquad t > 0. \tag{3.4}$$

One could simply take $q(t) = \sqrt{\log(t + 100)}$. Then (3.2)–(3.4) become requirements that M does not grow too swiftly.

Let

$$g_1(x, y) = g(x)\exp\{ye(x)\,M(|x|)\}.$$

We can write

$$f * g(x) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int f(y)\,g_1(x, y)\,dy,$$

$$I_2 = \int_{|y|>q(|x|)} f(y)\,\{g(x - y) - g_1(x, y)\}\,dy,$$

$$I_3 = \int_{|y|\leqslant q(|x|)} f(y)\,\{g(x - y) - g_1(x, y)\}\,dy.$$

We first note that

$$I_1 = g(x)\,Lf(e(x)\,M(|x|)) \geqslant g(x).$$

We shall show next that for sufficiently large values of x we have $|I_2| + |I_3| \leqslant g(x)$.

From Lemma 3.4, condition (3.2) and the superexponential decay of f we see that there exist numbers b and c such that

$$\begin{aligned}
|I_2| &\leqslant \int_{|y|>q(|x|)} |f(y)|\,g_1(x, y)\,dy \\
&\leqslant cg(x)\int_{|y|>q(|x|)} \exp\{-|y|(1 + h(|y|) + |y|\,M(|x|)\}\,dy \\
&\leqslant cg(x)\int_{|y|>q(|x|)} \exp\{-|y|\}\,dy \\
&\leqslant g(x)/3, \qquad |x| \geqslant b.
\end{aligned}$$

For I_3 we need the estimate

$$0 \geqslant |x| - |x-y| - ye(x) \geqslant -|y|^2/(2|x|). \tag{3.5}$$

The first inequality follows from Lemma 3.3. The second inequality follows from the estimate

$$\begin{aligned} |x-y| &= |x|(1 - 2xy|x|^{-2} + |y|^2|x|^{-2})^{1/2} \\ &\leqslant |x|\{1 - xy|x|^{-2} + |y|^2|x|^{-2}/2] \end{aligned}$$

Next, it follows from (3.3) and (3.5) that if $|y| < q(|x|)$ and x is sufficiently large, then

$$\begin{aligned} &\left| \int_{|x|}^{|x-y|} M(t)\,dt + ye(x)\,M(|x|) \right| \\ &\quad = \left| \int_{|x|}^{|x-y|} \{M(t) - M(|x|)\}\,dt + M(|x|)\,(ye(x) + |x-y| - |x|) \right| \\ &\quad \leqslant |M'(z)|\,q(|x|)^2 + M(|x|)\,q(|x|)^2/|x| \\ &\quad \leqslant 2(1 + M(|x|))\,q(|x|)^2/|x|\,. \end{aligned}$$

Here z is a number between $|x|$ and $|x-y|$, and we used condition (3.3) in the last step.

Thus we obtain, using condition (3.4), that

$$\begin{aligned} |I_3| &\leqslant \int_{|y| \leqslant q(|x|)} |f(y)|\,g(x) \exp\{ye(x)\,M(|x|)\} \\ &\qquad \times \left| 1 - \exp\left\{ -\int_{|x|}^{|x-y|} M(t)\,dt - ye(x)\,M(|x|) \right\} \right| dy \\ &\leqslant 4g(x) \int |f(y)|\,dy \exp\{q(|x|)\,M(|x|)\}\,(1 + M(|x|))\,q(|x|)^2/|x| \\ &\leqslant g(x)/3, \end{aligned}$$

for $|x|$ exceeding some number b'.

Combining the estimates for I_1, I_2, and I_3 we obtain

$$f * g(x) \geqslant g(x)/3 \geqslant 0,$$

for $|x| \geqslant \max(b, b')$, provided that M and q are *any* pair of functions satisfying (3.2), (3.3) and (3.4).

It remains to define M further to show $f * g(x) \geqslant 0$ for small values of $|x|$. To do this we first choose a number $a > 0$ such that

$$Lf(0) \geqslant 2 \int_{|y|>a} |f(y)|\,dy.$$

If we take $M(t) = 0$ for $0 \leqslant t \leqslant 2a'$ for some $a' \geqslant a$, and require only that $M(t) \geqslant 0$ elsewhere, then we have for $|x| \leqslant a'$ that

$$f * g(x) \geqslant \int_{|y-x|<2a} f(y)\,dy - \int_{|y-x|>2a} |f(y)|\,dy$$
$$\geqslant Lf(0) - 2\int_{|y|>a} |f(y)|\,dy \geqslant 0.$$

We now choose $a' = \max(a, b, b')$, $M(t) = 0$ for $0 \leqslant t \leqslant 2a'$, and satisfying (3.2), (3.3) and (3.4). By the preceding paragraph $f * g(x) \geqslant 0$ for $|x| \leqslant a'$, and earlier we had shown that $f * g(x) \geqslant 0$ for $|x| \geqslant \max(b, b')$. Thus $f * g(x) \geqslant 0$ for $|x| \geqslant \max(b, b')$. Thus $f * g(x) \geqslant 0$ on $\mathscr{R}^d$.

Example. Suppose that

$$f(x) = O\{\exp(-|x|^{1+\alpha})\}$$

for some $\alpha > 0$. Take $q(t) = \{\log(t + 100)\}^{1/(1+\alpha)}/2$,

$$\begin{aligned} M(t) &= 0, && 0 \leqslant t \leqslant a' \\ &= \{\log(t+100)/(a'+100)\}^{\alpha/(1+\alpha)}, && t > a', \end{aligned}$$

where a' depends upon f and α and $g(x) = \exp\{-\int_0^{|x|} M(t)\,dt\}$. The function g vanishes superexponentially, but substantially less swiftly than f.

Suppose now that we replace the underlying group $\mathscr{R}^d$ by $\mathscr{Z}^d$. With such obvious modifications as defining the Laplace transform by

$$Lf(x) = \sum_{n \in \mathscr{Z}^d} f(n)\, e^{nx},$$

the analogue of Theorem 3.2 is valid for this case also.

4. Compactly Supported Functions on R

Let $f \in \mathscr{F}_c$, i.e., f is a compactly supported real-valued L^1 function on $\mathscr{R}$. We define the *support interval* of f to be the closed convex hull of the support of f. Let $[A, B]$ denote the support interval of f. We say that f is ***boundary definite*** if there exists a number $\delta > 0$ such that f, or some L^1 equivalent, is non-negative on each of the intervals $[A, A + \delta]$, $[B - \delta, B]$.

Theorem 4.1. *Let $f \in \mathscr{F}_c(\mathscr{R})$, $f \not\equiv 0$. $\mathscr{P}(f, \mathscr{F}_c) \neq \varnothing$ if and only if the following two conditions hold:*

$$Lf(x) > 0 \text{ on } \mathscr{R}; \tag{4.1}$$

there exists a non-negative compactly supported function $\varphi \in L^1(\mathscr{R})$ such that $f * \varphi$ is boundary definite. (4.2)

Proof. Suppose first that there exists a compactly supported function $g \in \mathscr{P}(f)$. Then f and g have superexponential decay and by Theorem 3.2, $Lf > 0$ on $\mathscr{R}$. The function $f * g$ is clearly boundary definite.

Conversely, suppose (4.1) and (4.2) hold. For the while let us assume that f itself is boundary definite. There is no loss in generality in assuming that the support interval of f has the form $[-A, A]$.

Since $f \in \mathscr{F}_c(\mathscr{R})$ is of superexponential decay at ∞, the proof of Theorem 3.2 shows that there exists a function

$$g_0(x) = \exp\left\{-\int_0^{|x|} M(t)\, dt\right\}, \qquad x \in \mathscr{R},$$

where M is a non-decreasing non-negative and unbounded function on $\mathscr{R}^+$, such that $f * g_0$ is non-negative. We shall take

$$\begin{aligned} g(x) &= g_0(x), \qquad && |x| \leqslant RA, \\ &= 0, && |x| > RA, \end{aligned}$$

for some $R > 0$. It is clear that

$$\begin{aligned} f * g(x) &= f * g_0(x) > 0, \qquad && |x| < RA - A, \\ f * g(x) &= 0, && |x| > RA + A. \end{aligned}$$

We must prove that

$$f * g(x) \geqslant 0, \qquad RA - A \leqslant |x| \leqslant RA + A. \tag{4.3}$$

It clearly suffices to give the argument for the interval $[RA - A, RA + A]$.

The boundary definite condition implies that $f(t) \geqslant 0$ for $A - 2\delta \leqslant t \leqslant A$ for some $\delta > 0$. It follows that for $RA + A - 2\delta \leqslant x \leqslant RA + A$ we have

$$f * g(x) = \int_{x-A}^{AR} f(x - t)\, g(t)\, dt \geqslant 0$$

since both functions are non-negative in the overlapping region of support.

If $RA - A \leqslant x < RA + A - 2\delta$, then

$$\begin{aligned} f * g(x) = \int_{x-A}^{RA} f(x-t)\, g(t)\, dt &\geqslant \left\{\int_{x-A}^{x-A+\delta} - \int_{x-A+2\delta}^{RA}\right\} |f(x-t)|\, g(t)\, dt \\ &> g(x - A + \delta) \left\{\int_{A-\delta}^{A} f(t)\, dt - e^{-\delta M(x-A+\delta)} \int_{-A}^{A} |f(t)|\, dt\right\} \\ &\geqslant g(x - A + \delta) \left\{\int_{A-\delta}^{A} f(t)\, dt - e^{-\delta M(RA-2A)} \int_{-A}^{A} |f(t)|\, dt\right\}, \end{aligned}$$

which is non-negative provided only that

$$\exp\{\delta M(RA - 2A)\} \geqslant \left\{\int_{-A}^{A} |f(t)|\, dt\right\} \Big/ \left\{\int_{A-\delta}^{A} f(t)\, dt\right\}.$$

Since M is unbounded, there exists a number R for which the last inequality is satisfied. Thus $f * g \geqslant 0$ on $\mathscr{R}$ in case f itself is boundary definite.

If φ is a compactly supported non-negative L^1 function for which $f * \varphi$ is boundary definite, let $F = f * \varphi$. Then $LF > 0$ on $\mathscr{R}$, and we carry out the preceding argument for F in place of f and find a function $g \in \mathscr{P}(F, \mathscr{F}_c)$. The function $\varphi * g$ is compactly supported and belongs to $\mathscr{P}(f)$.

The remainder of this section considers the role of the boundary definite property. Simple examples show that it is not necessary that f itself be boundary definite in order that $\mathscr{P}(f, \mathscr{F}_c) \neq \varnothing$. On the other hand, we shall see that $Lf > 0$ on $\mathscr{R}$ is not a sufficient condition for $\mathscr{P}(f, \mathscr{F}_c) \neq \varnothing$. We begin by establishing bounds for Lf which must be satisfied in order that $\mathscr{P}(f) \neq \varnothing$.

THEOREM 4.2. *Let f fulfill the conditions of Theorem 4.1. Then there exist real numbers A and B such that*

$$\lim_{x\to\infty} x^{-1} \log Lf(x) = B, \qquad \lim_{x\to-\infty} x^{-1} \log Lf(x) = A. \tag{4.4}$$

Further, for any $\epsilon > 0$ there exists a number $c = c(\epsilon, f) > 0$ such that

$$Lf(x) \geqslant c\{e^{(A+\epsilon)x} + e^{(B-\epsilon)x}\}, \qquad x \in \mathscr{R}.$$

Proof. Let $g \in \mathscr{P}(f)$ have $[C, D]$ as its support interval. Since g is non-negative we clearly have

$$\lim_{x\to\infty} x^{-1} \log Lg(x) = D, \qquad \lim_{x\to-\infty} x^{-1} Lg(x) = C. \tag{4.5}$$

It follows from the Titchmarsh convolution theorem [12, Ch. VI, Section 4] that the support interval of $f * g$ is $[A + C, B + D]$. Since $f * g$ is also non-negative, we see from (4.5) that

$$B + D = \lim_{x\to\infty} \frac{\log L(f * g)(x)}{x} = \lim_{x\to\infty} \frac{\log Lf(x)}{x} + D,$$

$$A + C = \lim_{x\to-\infty} \frac{\log L(f * g)(x)}{x} = \lim_{x\to-\infty} \frac{\log Lf(x)}{x} + C.$$

Thus (4.4) is true. Also, Lf is continuous and positive on $\mathscr{R}$. Thus the lower bound for Lf holds on $\mathscr{R}$.

Example. We shall now give an example of a compactly supported function f with $Lf > 0$ on $\mathscr{R}$ but such that $\mathscr{P}(f, \mathscr{F}_c) = \varnothing$. Let $\{a_\nu\}_1^\infty$ be a sequence of positive numbers satisfying

$$\sum_{\nu=p+1}^{\infty} a_\nu < a_p/(3e), \qquad p = 1, 2,.... \tag{4.6}$$

Let $\{k_\nu\}$ be an increasing sequence of positive integers such that

$$\sum_{\nu=1}^{p-1} a_\nu \exp\{1 - 2^{k_p - k_{p-1}}\} < \frac{a_p}{3}, \qquad p = 2, 3,\ldots \tag{4.7}$$

For example, we could take $a_p = 10^{-p}$ and $k_p = 3^p$.

Now let

$$\begin{aligned} f_\nu(x) &= 1, \qquad 0 \leqslant x \leqslant 1 - 2^{-k_\nu}, \\ &= 0, \qquad \text{elsewhere,} \end{aligned}$$

and

$$f(x) = \sum_{\nu=1}^{\infty} a_\nu (-1)^\nu f_\nu(x).$$

An easy computation gives

$$xLf(x) = \sum_1^\infty a_\nu(-1)^\nu \exp(x - x\,2^{-k_\nu}) - \sum_1^\infty (-1)^\nu a_\nu .$$

We are going to find the sign of $xLf(x)$ at $x = 2^{k_p}$. It follows from (4.7) that

$$\begin{aligned} \left| \sum_1^{p-1} (-1)^\nu a_\nu \exp(2^{k_p} - 2^{k_p - k_\nu}) \right| &\leqslant \exp(2^{k_p} - 1) \sum_1^{p-1} a_\nu \exp(1 - 2^{k_p - k_\nu}) \\ &\leqslant \frac{a_p}{3} \exp(2^{k_p} - 1). \end{aligned}$$

Relation (4.6) gives

$$\left| \sum_{p+1}^\infty (-1)^\nu a_\nu \exp(2^{k_p} - 2^{k_p - k_\nu}) \right| \leqslant \exp(2^{k_p} - 1)\, a_p/3,$$

and thus

$$\begin{aligned} &| 2^{k_p} Lf(2^{k_p}) - (-1)^p a_p \exp(2^{k_p} - 1)| \\ &\qquad \leqslant 2a_p \exp(2^{k_p} - 1)/3 + a_1 < 3a_p \exp(2^{k_p} - 1)/4. \end{aligned}$$

It follows that the sign of $xLf(x)$ alternates as x runs through the sequence $\{2^{k_p}\}_{p=1}^\infty$. Thus there exists an increasing unbounded sequence $\{b_p\}_1^\infty$ such that $Lf(b_p) = 0$ for $p = 1, 2,\ldots$.

Let g be a non-negative function supported on $[-\delta, \delta]$ for some $\delta \in (0, 1/2)$. Further suppose $Lg \geqslant 1$ on $\mathscr{R}$. If we set $F = g + f * f$, then $LF = Lg + (Lf)^2 \geqslant 1$. We show that

$$\liminf_{x\to\infty} x^{-1} \log LF(x) < \limsup_{x\to\infty} x^{-1} LF(x).$$

We have

$$\liminf_{x\to\infty} (\log LF(x))/x \leqslant \lim_{p\to\infty} (\log LF(b_p))/b_p \leqslant \delta,$$

$$\begin{aligned} \limsup_{x\to\infty} (\log LF(x))/x &\geqslant \lim_{p\to\infty} (\log LF(2^{k_p}))/2^{k_p} \\ &= 2 + \lim_{p\to\infty} 2^{-k_p} \log a_p = 2. \end{aligned}$$

The last equality is deduced by noting that (4.7) implies

$$a_1 \exp(-2^{k_p-k_{p-1}}) < a_p\,, \qquad p = 2, 3, \ldots.$$

Thus $x^{-1} \log LF(x)$ has no limit as $x \to \infty$, and it follows from Theorem 4.2 that $\mathscr{P}(F, \mathscr{F}_c) = \varnothing$.

We shall now consider the Laplace transform in the complex plane. The following results suggest that the complete solution of our problem for compactly supported functions is to be found here. We begin with an extension of the observation that the Laplace transform of a positive function is positive on $\mathscr{R}$.

LEMMA 4.3. *Let h be a non-trivial non-negative valued function on $\mathscr{R}$ with support interval $[A, B]$. Then $Lh(z) \neq 0$ for $|\operatorname{Im} z| \leqslant \pi/(B - A)$.*

Proof. Let $\beta = (B + A)/2$ and $z = x + iy$ with x, y real. We have

$$\operatorname{Re}\{e^{-iy\beta} Lh(z)\} = \int_A^B h(t)\, e^{xt} \cos(yt - y\beta)\, dt > 0$$

since $|y|\ |t - \beta| < \pi/2$ for $A < t < B$.

Actually, we can say rather more with a little more effort.

THEOREM 4.4. *Let $f \in \mathscr{F}_c$, $f \not\equiv 0$, be boundary definite. There exists a positive number B and a non-negative function q satisfying $q(x) \to \infty$ as $|x| \to \infty$ so that $Lf(z) \neq 0$ on*

$$D = \{z = x + iy : |x| \geqslant B,\ |y| \leqslant q(x)\}.$$

Proof. We can assume without loss of generality that the support interval of f is $[-A, A]$. By hypothesis there exists a positive number α such that

$$f(x) \geqslant 0, \qquad A - \alpha \leqslant x \leqslant A.$$

Let δ and η be positive numbers satisfying

$$\delta + \eta \leqslant \alpha, \tag{4.8}$$

$$b \overset{\text{def}}{=} \tfrac{1}{3} \int_{A-\delta}^{A} f(t)\, dt \geqslant \int_{A-\delta-\eta}^{A-\delta} f(t)\, dt. \tag{4.9}$$

Let

$$D(\delta, \eta) = \{z = x + iy \in \mathscr{C} : x \geqslant \eta^{-1} \log(3M/b), \qquad |y| \leqslant \pi/(3\delta)\}.$$

Here $M = \int_{-A}^{A} |f(t)|\, dt$. We claim that

$$Lf(z) \neq 0, \qquad z \in D(\delta, \eta). \tag{4.10}$$

To prove (4.10) we note that if $x > 0$,

$$\left| \int_{-A}^{A-\delta-\eta} f(t)\, e^{zt}\, dt \right| \leqslant Me^{(A-\delta-\eta)x},$$

$$\left| \int_{A-\delta-\eta}^{A-\delta} f(t)\, e^{zt}\, dt \right| \leqslant be^{(A-\delta)x},$$

$$\begin{aligned} \left| \int_{A-\delta}^{A} f(t)\, e^{zt}\, dt \right| &= \left| \int_{A-\delta}^{A} f(t)\, e^{xt+iy(t-A)}\, dt \right| \\ &\geqslant \int_{A-\delta}^{A} f(t)\, e^{x} \cos y(t-A)\, dt \geqslant \frac{3b}{2}\, e^{x(A-\delta)}, \end{aligned}$$

assuming that $|\, y\delta \,| \leqslant \pi/3$.

In $D(\delta, \eta)$ we have $x > 0$, $|\, y\delta \,| \leqslant \pi/3$, and $Me^{-\eta x} \leqslant b/3$. Thus we have

$$\left| \int_{-A}^{A-\delta} f(t)\, e^{zt}\, dt \right| < e^{(A-\delta)x}\, 4b/3 < \left| \int_{A-\delta}^{A} f(t)\, e^{zt}\, dt \right|$$

for all $z \in D(\delta, \eta)$. It follows from Rouché's Theorem that $Lf(z)$ and $\int_{A-\delta}^{A} f(t) e^{zt}\, dt$ have the same number of zeros in $D(\delta, \eta)$, namely none.

Let $\alpha \geqslant \delta_1 > \delta_2 > \cdots$ be a sequence converging to zero. For each δ_k let η_k be the largest value of η such that (4.8) and (4.9) both hold. Let b_k be the value b associated with δ_k and let

$$x_k = \eta_k^{-1} \log(3M/b_k), \qquad y_k = \pi/(3\delta_k).$$

As $\delta_k \downarrow 0$, we have $b_k \downarrow 0$, $\eta_k \to 0$, $x_k \to \infty$, $y_k \uparrow \infty$. Take $B = \min_{k\geqslant 1} x_k$. Given $x \geqslant B$, we choose m such that $x_m \leqslant x < x_n$ for all $n > m$ and define $q(x) = y_m$. It is clear that $Lf(x + iy) \neq 0$ for $x \geqslant B$ and $|\, y \,| \leqslant q(x)$. A similar argument works for $x \to -\infty$.

Corollary 4.5. *Let $f \in F_c$ and assume that $\mathscr{P}(f, \mathscr{F}_c) \neq \varnothing$. Then there exists a region D of the type described in the preceding theorem such that $Lf(z) \neq 0$ in D.*

Proof. Let g be a non-negative function in F_c such that $f * g \geqslant 0$ on $\mathscr{R}$. By Theorem 4.4 there exists a region D on which $L(f * g)$ is non-zero. Since $L(f * g) = (Lf)(Lg)$ and Lg is an entire function, it follows that $Lf(z) \neq 0$ for $z \in D$.

Remark 1. It is clear from the proof of Lemma 4.3 and Corollary 4.5 that if the support interval of $f * g$ is short, the zero-free region of Lf is wide.

Remark 2. Theorem 4.2 suggests that the condition $L(|\, f \,|) = O(Lf)$ on $\mathscr{R}$ may be sufficient to guarantee the existence of a function $g \in \mathscr{P}(f, \mathscr{F}_c)$. We have not been able to prove this conjecture. The following example shows that the condition is not necessary.

Let f be an even function on $\mathscr{R}$ which is defined as follows:

$$\begin{aligned} f(x) &= 1, && 0 \leqslant |x| \leqslant \tfrac{1}{3}, \\ &= -1, && 1 - e^{-n} - e^{-2n} \leqslant |x| \leqslant 1 - e^{-n}, \quad n = 1, 2,..., \\ &= 1, && 1 - e^{-n} \leqslant |x| \leqslant 1 - e^{-n} + e^{-2n}, \quad n = 1, 2,..., \\ &= 0, && \text{other values of } x. \end{aligned}$$

Let $\chi_{[-1,1]}$ be the characteristic function of the interval $[-1, 1]$. Then $f * \chi_{[-1,1]} \geqslant 0$ on $\mathscr{R}$. By considering the largest term in the series we have for some $c > 0$ that

$$L\,|f|\,(x) > cx^{-2}e^{x}, \qquad x \to \infty.$$

Also, simple estimates show that

$$Lf(x) < c'x^{-3}e^{x} \log x, \qquad x \to \infty,$$

for some $c' > 0$. Thus $Lf = o(L\,|f|)$ at ∞.

5. The Size of the Support Interval of $g \in \mathscr{P}(f, \mathscr{F}_c(\mathscr{Z}))$

Given $f \in \mathscr{F}_c(\mathscr{Z})$, $f \not\equiv 0$, we define the *support interval* of f to be $[M, N]$ if $f(M)f(N) \neq 0$ and $f(n) = 0$ for all $n < M$ and $n > N$. The length $S(f)$ of the support interval is $N - M + 1$.

Given a non-trivial function $f \in \mathscr{F}_c(\mathscr{Z})$ with $Lf > 0$ on $\mathscr{R}$, we ask: what is the minimal value of $S(g)$ as g ranges over elements of $\mathscr{P}(f, \mathscr{F}_c)$? We set

$$M(f) = \min\{S(g) : g \in \mathscr{P}(f)\}.$$

The problem of characterizing $f \in \mathscr{F}_c(\mathscr{Z})$ with $\mathscr{P}(f) \neq \varnothing$ was solved by Poincaré [7] and Meissner [6]. Consider the polynomial $P_f(x) = \sum f(n)x^n$. It is clear that a necessary condition for $\mathscr{P}(f, \mathscr{F}_c) \neq \varnothing$ is that $P_f(x) > 0$ for $x > 0$.

We shall sketch the method used in [6, 7] for proving the converse proposition. The ideas introduced will be helpful to us presently. If $P_f(x) > 0$ for $x > 0$, we can express P_f in an essentially unique way as a product of linear factors with negative roots, real quadratic factors with non-real roots and powers of x. The linear factors, the quadratic factors with positive coefficients and the powers of x need no further treatment. Each of the remaining factors has the form $P(x) = Ax^2 - Bx + C$, where $A, B, C > 0$ and $B^2 < 4AC$.

As we shall see presently, for each such factor $P(x)$ we can find a polynomial $Q(x)$ with positive coefficients and such that $P(x)\,Q(x)$ has positive coefficients. The product of these $Q(x)$'s gives a polynomial $Q_f(x)$ with positive coefficients such that $P_f(x)\,Q_f(x)$ has positive coefficients.

It is convenient to introduce a normalization. Given $P(x)$ as above, let

$$\alpha = \arccos\{B/(2\sqrt{(AC)})\}, \qquad 0 < \alpha < \pi/2,$$

and let

$$P_\alpha(x) = P(x\sqrt{(C/A)})/C = x^2 - 2x\cos\alpha + 1$$

denote a normalized form of $P(x)$. Let f denote the associated function on $\mathscr{Z}$ defined by $f_\alpha(n) = 0$, $n \leqslant -1$ or $n \geqslant 3$ and

$$\{f_\alpha(0), f_\alpha(1), f_\alpha(2)\} = \{1, -2\cos\alpha, 1\}.$$

Consideration of the Poisson kernel

$$(1 - 2x\cos\alpha + x^2)^{-1} = \sum_{n=0}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} x^n$$

or the homogeneous second order difference equation [5, pp. 123–124]

$$f_\alpha * \psi(n) = \psi(n+1) - 2\psi(n) + \psi(n-1) + (2 - 2\cos\alpha)\psi(n) = 0$$

suggests that we consider

$$\begin{aligned} \psi(n) &= \sin n\alpha, \qquad && 0 < n < \pi/\alpha, \\ &= 0, && n \leqslant 0 \quad \text{or} \quad n \geqslant \pi/\alpha. \end{aligned} \tag{5.1}$$

It is easy to see that $f_\alpha * \psi \geqslant 0$ on $\mathscr{Z}$.

Remark. The choice $\psi(n) = \binom{N}{n}$ for a sufficiently large positive integer $N = N(\alpha)$ gives another solution of $f_\alpha * \psi \geqslant 0$. (W. L. Putnam Math. Examination question $A4$, December 1971 [8].)

We shall now find $M(f)$ when $Lf > 0$ and $S(f) = 3$. It is convenient to normalize the problem. For some N we have $f(N) > 0$ and $f(N+2) > 0$. This follows from considering Lf near $+\infty$ and $-\infty$. We may assume that $f(N+1) < 0$, for otherwise the choice $g(0) = 1$, $g(n) = 0$ for all $n \neq 0$ is obviously minimal. By translating f we may assume $N = 0$ and by making the polynomial normalization described above, we can assume that $f(0) = f(2) = 1$ and $-2 < f(1) = -2\cos\alpha < 0$ for some $\alpha \in (0, \pi/2)$.

THEOREM 5.1. *Let f be as above. Then*

$$\begin{aligned} M(f) = -\left[1 - \frac{\pi}{\alpha}\right] &= [\pi/\alpha], \qquad && \textit{if} \quad \pi/\alpha \notin \mathscr{Z}^+, \\ &= (\pi/\alpha) - 1, && \textit{if} \quad \pi/\alpha \in \mathscr{Z}^+. \end{aligned}$$

In the latter case the extremal $g \in \mathscr{P}(f)$ is determined uniquely to within multiplication by a positive constant and translation by integers by the formula

$$(Lg)(\log x) = \{x^{\pi/\alpha} + 1\}/\{x^2 - 2x\cos\alpha + 1\}.$$

For example, if $\pi/\alpha = 4$, then $M(f) = 3$,

$$(Lf)(\log x) = 1 - \sqrt{2}x + x^2$$

and g, unique up to multiples and translations, is given by

$$(Lg)(\log x) = 1 + \sqrt{2}x + x^2.$$

The product satisfies

$$\{L(f * g)\} (\log x) = 1 + x^4.$$

Proof. We shall show that $M(f) = S(\psi)$, where ψ is the function defined by (5.1). Let g be any function in $\mathscr{P}(f)$ with $S(g) = N$, say. There is no loss in generality in assuming that the support interval of g is $[1, N]$ and that $g(1) = \sin \alpha$. We shall show that

$$g(n) \geqslant \sin n\alpha, \qquad 1 \leqslant n \leqslant S(\psi), \tag{5.2}$$

and hence $S(g) \geqslant S(\psi)$.

Let k be the least positive integer for which $g(n) \neq \sin n\alpha$. Clearly $k \geqslant 2$. On the other hand, we may suppose that $k \leqslant S(\psi)$, for otherwise (5.2) holds at once. The relation $f * g \geqslant 0$ gives

$$g(k) - 2 \cos \alpha \sin(k - 1)\alpha + \sin(k - 2)\alpha \geqslant 0$$

or $g(k) \geqslant \sin k\alpha$. By the definition of k, strict inequality holds.

Now we shall show that $g(n) > \sin n\alpha$ for $k \leqslant n \leqslant S(\psi)$. For $n \geqslant 0$, define $s(n) = \sin n\alpha$, and note that $(f * s)(n) = 0$ for $n \geqslant 2$. Thus

$$\{f * (g - s)\} (n) \geqslant 0 \qquad \text{for} \quad n \geqslant 2. \tag{5.3}$$

Also, we define a sequence $\{a_\nu\}$ by setting $a_0 = 1$, $a_1 = 2 \cos \alpha$ and for $\nu \geqslant 1$, $a_{\nu+1} = 2 \cos \alpha - (1/a_\nu)$. One verifies by induction that

$$a_\nu = \sin(\alpha\nu + \alpha)/\sin \alpha\nu \qquad \text{for} \quad \nu \geqslant 1. \tag{5.4}$$

Now (5.3) implies that

$$(g - s)(k + 1) \geqslant 2 \cos \alpha(g - s)(k) = a_1(g - s)(k) > 0;$$
$$(g - s)(k + 2) \geqslant 2 \cos \alpha(g - s)(k + 1) - (g - s)(k) \geqslant a_2(g - s)(k + 1) > 0,$$

provided that $a_2 > 0$, and generally

$$\begin{aligned}(g - s)(k + \nu) &\geqslant 2 \cos \alpha(g - s)(k + \nu - 1) - (g - s)(k + \nu - 2) \\ &\geqslant a_\nu(g - s)(k + \nu - 1) > 0\end{aligned} \tag{5.5}$$

provided that $a_i > 0$, $1 \leqslant i \leqslant \nu$. We see from (5.4) that the $a_i > 0$ provided that $i + 1 < \pi/\alpha$, i.e., $i + 1 \leqslant S(\psi)$. Thus the ν in (5.5) can be taken as large as $S(\psi) - 1$. Recalling that $k \geqslant 2$, we see that $g(n) > \sin n\alpha$ for $k \leqslant n \leqslant S(\psi)$.

Now suppose $\pi/\alpha \in \mathscr{Z}$ and $g(k)/\sin \alpha \neq \psi(k)$ for some k in $[1, (\pi/\alpha) - 1]$. The foregoing argument shows that

$$g(\pi/\alpha)/\sin \alpha > \psi(\pi/\alpha) = 0,$$

and hence $S(g) > S(\psi)$. Thus the function $g \in \mathscr{P}(f)$ satisfying $S(g) = M(f)$ is unique up to positive multiples and integer translations if $\pi/\alpha \in \mathscr{Z}$.

The problem of finding $M(f)$ for a function f satisfying $3 < S(f) < \infty$ is presently unsolved. The method of proving Theorem 5.1 offers the upper estimate

$$M(f) \leqslant \sum_{k=1}^{p} S(\psi_k).$$

where p is the number of quadratic factors in f which need multipliers. This This estimate may be very poor, as the following example shows. Let $N \geqslant 3$ be an odd integer. Let $f(n) = (-1)^n$, $0 \leqslant n \leqslant N - 1$, and $f(n) = 0$ for all other n. Then

$$\begin{aligned} Lf(\log x) &= x^{N-1} - x^{N-2} + x^{N-3} - \cdots + 1 \\ &= \prod_{j=0}^{(N-3)/2} \left(x - e^{(2j+1)\pi i/N}\right)\left(x - e^{-(2j+1)\pi i/N}\right), \end{aligned}$$

and we obtain the bound $M(f) = O(N \log N)$. In fact $M(f) = 2$, since the identity

$$(x^{N-1} - x^{N-2} + x^{N-3} - \cdots + 1)(x + 1) = x^N + 1$$

shows that we can take $g(0) = g(1) = 1$ and $g(n) = 0$ elsewhere.

Theorem 5.1 showed an instance in which there existed a function $g \in \mathscr{P}(f)$ so that $(f * g)(n) = 0$ for all but two values of n; namely if $\alpha = \pi/N$ for some integer N. Here we shall show that this phenomenon can occur for no other values of α.

THEOREM 5.2. *Let* $f\colon \mathscr{Z} \to \mathscr{R}$ *satisfy* $S(f) = 3$, $f(0) = f(2) = 1$, *and* $-2 < f(1) = -2 \cos \alpha < 0$ *for some* $\alpha \in (0, \pi/2)$. *Suppose there exists a function* $g \in \mathscr{P}(f)$ *such that* $(f * g)(n) = 0$ *for all but two values of* n. *Then* $\alpha = \pi/N$ *where* $N = S(g) + 1$. *Also* g *is the essentially unique element of* $\mathscr{P}(f)$ *having minimal support interval.*

Proof. Let

$$G(x) = (Lg)(\log x) = g(N - 2)\, x^{N-2} + g(N - 3)\, x^{N-1} + \cdots + g(1)\, x + g(0).$$

Then by hypothesis

$$(x^2 - 2x \cos \alpha + 1)\, G(x) = ax^N + b$$

where $a = g(N-2) > 0$ and $b = g(0) > 0$. We can symmetrize and normalize the functions by taking

$$(x^2 - 2x \cos \alpha + 1) \left\{ \frac{G(x) + x^{N-2}G(1/x)}{a+b} \right\} = x^N + 1.$$

Let us assume that G has been redefined to reflect this symmetrization and normalization:

$$(x^2 - 2x \cos \alpha + 1)\, G(x) = x^N + 1.$$

The nth coefficient of $G(x)$ is seen to satisfy

$$g(n) = \sin(\alpha n + \alpha)/\sin \alpha, \qquad 0 \leqslant n \leqslant N-2.$$

We have

$$g(0) = 1 = g(N-2) = \sin(N\alpha - \alpha)/\sin \alpha.$$

Also, since $S(g) \geqslant 2$, we have $N - 2 \geqslant 1$, and hence $N\alpha - \alpha \neq \alpha$. Thus there exist numbers $\alpha \neq (N-1)\alpha$ for which $\sin \alpha = \sin(N\alpha - \alpha)$. Finally, since the values of g are all non-negative, we must have $(N-1)\alpha < \pi$. The graph of sine shows that α and $(N-1)\alpha$ are located symmetrically with respect to $\pi/2$, i.e., $N\alpha = \alpha + (N-1)\alpha = \pi$.

The last assertion of the theorem follows immediately from Theorem 5.1.

6. Compactly Supported Functions in $\mathscr{Z}^d$ and $\mathscr{R}^d$, $d \geqslant 2$

Let us first consider $\mathscr{F}_0(\mathscr{Z}^d)$. If $f \in \mathscr{F}_0(\mathscr{Z}^d)$ is given, we can without loss of generality assume that the support of f is contained in $(\mathscr{Z}^+)^d$. With f, we associate the polynomial

$$P_f(x) = \sum_n f(n)\, x^n, \qquad x \in \mathscr{R}^d.$$

The result of Poincaré [7] and Meissner [6] has been generalized to polynomials in more than one variable by Hardy *et al.* ([4, Section 2.24]).

Theorem A. *Let F be a homogeneous real polynomial in $d+1$ variables, $d \geqslant 1$. If $F(x) > 0$, $x_i \geqslant 0$, $i = 1, 2,..., d+1$, $\sum_1^{d+1} x_i > 0$, then there exist homogeneous polynomials G and H with positive coefficients such that $F = G/H$. Furthermore, we can take $H(x) = (x_1 + x_2 + \cdots + x_{d+1})^p$ for some positive integer p.*

It is easy to transform the polynomial P_f of d variables into a homogeneous polynomial of $d + 1$ variables. Thus Theorem A solves completely our problem in the class $\mathscr{F}_0(\mathscr{Z}^d)$.

Let us now consider $\mathscr{F}_c(\mathscr{R}^d)$. The case $d = 1$ has been discussed in Section 4. If $f \in \mathscr{F}_0(\mathscr{R}^d)$, let supp$(f)$ be the support of f and let $\Omega(f)$ be the closed convex hull of the support of f.

Let π be the class of all hyperplanes of support to $\Omega(f)$. If $\delta > 0$ and $L \in \pi$ are given, we define

$$E(L, \delta) = \{y \in \operatorname{supp}(f)\colon \text{dist. } (y, L) \leqslant \delta\}.$$

We say that f is *boundary definite* if there exists a positive number $\delta = \delta(f)$ such that f or some L^1 equivalent is non-negative on the set $\bigcup_{L \in \pi} E(L, \delta)$. If f is boundary definite, we claim that

$$I(L, \eta) = \int_{E(L,\eta)} f(y)\, dy > 0 \tag{6.1}$$

for all $L \in \pi$ and all $\eta \in (0, \delta]$,

$$\alpha(\eta) = \inf_{L \in \pi} I(L, \eta) > 0, \qquad \eta \in (0, \delta]. \tag{6.2}$$

Proof of (6.1). If $I(L, \eta) = 0$ for some L and η, L is clearly not a hyperplane of support of $\Omega(f)$.

Proof of (6.2). If (6.2) is incorrect, there exists $\eta_0 \in (0, \delta]$ and a sequence $\{L_\nu\}$ in π such that $I(L_\nu, \eta_0) \to 0$, $\nu \to \infty$. For each ν, there exists $x_\nu \in \operatorname{supp}(f) \cap L_\nu$. We choose a subsequence $\{\nu_k\}$ in such a way that $x_{\nu_k} \to x_0 \in \operatorname{supp}(f)$, $k \to \infty$, and that as $k \to \infty$, the normals of the hyperplanes L_{ν_k} tend to a fixed direction orthogonal to a hyperplane of support L_0 through x_0. Let

$$F(p) = \bigcap_{k=p}^{\infty} E(L_{\nu_k}, \eta_0).$$

Then

$$I(L_0, \eta_0) = \lim_{p \to \infty} \int_{F(p)} f(y)\, dy \leqslant \lim_{p \to \infty} I(L_{\nu_p}, \eta_0) \doteq 0.$$

Thus L_0 is not a hyperplane of support. The contradiction shows that (6.2) is true.

THEOREM 6.1. *Let $f \in \mathscr{F}_c(\mathscr{R}^d)$, $f \not\equiv 0$. $\mathscr{P}(f, \mathscr{F}_c) \neq \varnothing$ if and only if the following two conditions hold.*

$$Lf(x) > 0 \text{ on } \mathscr{R}^d. \tag{6.3}$$

There exists a non-negative compactly supported function on $L^1(\mathscr{R}^d)$ such that $f * \varphi$ is boundary definite. (6.4)

Proof. For the first part of the proof, we refer to the corresponding part of the proof of Theorem 4.1. It remains to construct a multiplier g when f itself is boundary definite and (6.3) holds.

Let us assume that $\operatorname{supp}(f) \subset \{x \in \mathscr{R}^d: |x| \leqslant A\}$ for some number $A > 0$ and that the origin has been chosen in such a way that it belongs to $\Omega(f)$ and such that the smallest distance m from the origin to a hyperplane of support of $\Omega(f)$ is positive.

As in the proof of Theorem 4.1, we consider a function

$$g_0(x) = \exp\left\{- \int_0^{|x|} M(t)\, dt\right\}, \qquad x \in \mathscr{R}^d,$$

where $M(t) \uparrow \infty$ as $t \to \infty$, which is such that $f * g_0 \geqslant 0$. We shall take

$$\begin{aligned} g(x) &= g_0(x), \qquad & |x| \leqslant RA, \\ &= 0, & |x| > RA, \end{aligned}$$

for some $R > 0$. It suffices to prove that

$$f * g(x) \geqslant 0,\ RA - A \leqslant |x| \leqslant RA + A. \tag{6.5}$$

The boundary definite condition implies that $f(x) \geqslant 0$ for all points $x \in \operatorname{supp}(f)$ whose distance to $\operatorname{supp}(f) \cap \partial\Omega(f)$ is at most $4\eta > 0$, say, where we choose η in such a way that $\eta < m/2$. If, for given x, the distances from all points in $\mathscr{I}_R(x, f) = \{y \in \mathscr{R}^d : |x - y| = RA\} \cap \Omega(f)$ to $\partial\Omega(f)$ are at most 4η, then $f * g(x) \geqslant 0$ since both functions are non-negative in the overlapping region of support.

In the remaining case, there are points in $\mathscr{I}_R(x, f)$ whose distance to $\{y \in \mathscr{R}^d : |x - y| = RA\}$ exceeds 4η and for R large

$$H(x) = \operatorname{supp}(f) \cap \{y \in \mathscr{R}^d : |x - y| \leqslant RA\} \neq \varnothing.$$

Arguing as in the proof of Theorem 3.2, we see that

$$\begin{aligned} &\int_{H(x)} f(y)\, g(x - y)\, dy \\ &\quad \geqslant \int_{H(x)} f(y)\, g_1(x, y)\, dy - \int_{\operatorname{supp}(f)} |f(y)|\, (g_1(x, y) - g(x - y))\, dy \\ &\quad = J_1 - J_2, \end{aligned}$$

where $|J_2| \leqslant 2g(x)/3$, provided that $|x|$ is large enough. Since $|x| \geqslant A(R-1)$, this is correct if R is large enough.

Consider now the hyperplane of support $L(x)$ of $\Omega(f)$ orthogonal to x. If R is large, there exists $z \in \operatorname{supp} f \cap L(x)$ such that $|z - x| \leqslant RA - 3\eta$. Thus f is non-negative in $E(L(x), 3\eta)$ and the distance from a point in $E(L(x), \eta)$ to the closure of the set $\{y \in \mathscr{R}^d : f(y) < 0\}$ is at least 2η. Let $m(x)$ be the distance from the origin to $L = L(x)$. Then, using (6.2) and the fact that $ye(x) \geqslant m(x) - \eta$, $y \in E(L, \eta)$, we see that

$$J_1 \geqslant g(x) \left\{ \int_{E(L,\eta)} f(y)\, dy \exp\{(m(x) - \eta)\, M(|\, x\,|)\} \right.$$
$$\left. - \int_{\operatorname{supp}(f)} |f(y)|\, dy \exp\{(m(x) - 2\eta)\, M(|\, x\,|)\} \right\}$$
$$\geqslant g(x) \exp\{(m - \eta)\, M(|\, x\,|)\}\, \alpha(\eta)/2,$$

provided that $|\, x\,|$ and thus $M(|\, x\,|)$ are large enough. Adding up, we obtain in this case

$$\int_{H(x)} f(y)\, g(x - y)\, dy \geqslant g(x)\, \{\alpha(\eta) \exp\{(m/2)\, M(|\, x\,|)\}/2 - \tfrac{2}{3}\} > 0.$$

If R and thus also $|\, x\,|$ are large enough, all our estimates are valid and we have proved Theorem 6.1.

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